[Surendra Sharma
PUL078BCE178]

Initial value problem
→ Euler
→ RK

Simple Euler's Method

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$h \to$ step size

$f(x,y) \to$ slope of tangent.

Modified Euler's Method

same in each step.

$$(y_1)^n = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n-1)})\right]$$

$(y_1)^n \to n^{th}$ refinement of y_1

y_0 & $x_0 \to$ initial guess values / 'or' final output from previous step.

$x_1 \to x_0 + h$

$f(x,y) \to$ slope $= dy/dx$

RK method

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0, \quad y(x_0+h) = ?$$

$$y(x_0+h) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \frac{h^4}{4!}y^{iv}(x_0) + \ldots$$

RK-1

RK-2

RK-3

RK-4

- RK-1 method:

$$y(x_0+h) = y(x_0) + h \cdot y'(x_0)$$

- RK-2 method

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0)$$

Like modified Euler

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_0 + \underbrace{h f(x_0, y_0)}_{K_1})\right]$$

working rule,

$$K_1 = h f(x_0, y_0)$$
$$K_2 = h f(x_0 + h, y_0 + K_1)$$
$$K = (K_1 + K_2)/2$$
$$y_1 = y_0 + K$$

Similarly, $y_2 = y_1 + K$

RK-4 method

Working rule,

$$k_1 = h f(x_0, y_0) \quad \} \text{ NO } \div 2$$

$$k_2 = h f(x_0 + h/2, y_0 + \frac{k_1}{2}) \quad \} \div 2$$

$$k_3 = h f(x_0 + h/2, y_0 + \frac{k_2}{2}) \quad \} \div 2$$

$$k_4 = h f(x_0 + h, y_0 + k_3) \quad \} \text{ NO } \div 2$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \quad \} \text{ weighed avg.}$$

$$y_1 = y_0 + k$$

Given, $x_0, y_0, h, \frac{dy}{dx} = f(x, y)$

System of first order differential equation

(1 independent variable but can have any no. of dependent var)

$$\frac{dy}{dx} = f(x, y, z), \quad y(x_0) = y_0$$

$$\frac{dz}{dx} = g(x, y, z), \quad z(x_0) = z_0$$

Given x_0, y_0, h, z_0

Let, $\frac{dy}{dx} = f(x, y, z) = z$

find corres. $\frac{dz}{dx}$ i.e. $\frac{d^2y}{dx^2}$ in term of x, y, z

RK1-method

$$y_1 = y_0 + h f(x_0, y_0, z_0)$$

$$z_1 = z_0 + h g(x_0, y_0, z_0)$$

RK 2-method

$$k_1 = h f(x_0, y_0, z_0)$$

$$k_2 = h f(x_0 + h, y_0 + k_1, z_0 + l_1)$$

$$k = \frac{k_1 + k_2}{2}$$

$$y_1 = y_0 + k$$

$$l_1 = h g(x_0, y_0, z_0)$$

$$l_2 = h g(x_0 + h, y_0 + k_1, z_0 + l_1)$$

$$l = \frac{l_1 + l_2}{2}$$

$$z_1 = z_0 + l$$

RK 4-method

$$k_1 = h f(x_0, y_0, z_0)$$

$$k_2 = h f(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$$

$$k_3 = h f(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$$

$$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k = (k_1 + 2k_2 + 2k_3 + k_4)/6$$

$$y_1 = y_0 + K$$

$$l_1 = h g(x_0, y_0, z_0)$$

$$l_2 = h g(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$$

$$l_3 = h g(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$$

$$l_4 = h g(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l = (l_1 + 2l_2 + 2l_3 + l_4)/6$$

$$z_1 = z_0 + K$$

Boundary value problem (BVP)

① Finite difference method

$\rightarrow y'_x \approx \frac{y(x+h) - y(x)}{h}$ → F/W difference approximation.
(x+h − x = h से divide)

$\rightarrow y'(x) \approx \frac{y(x) - y(x-h)}{h}$ → B/W difference approximation.
(x − (x−h)) = h से divide

$\rightarrow y'(x) \approx \frac{y(x+h) - y(x-h)}{2h}$ → central difference approx.
(Best of 3 approx as neglects power of h^2)
(x+h − (x−h)) = 2h से divide

$\rightarrow y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$ → second order approximation formula.

In Notation,

$$y_i'' \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

* step size - must be given (directly or indirectly)

Working rule,

→ Given differential equation is converted into regular system (i^{th} coordinate) by central diff. formula and second order approx. formula.
(To find governing eqn)

→ According to no. of steps, for different values of i equations were developed & solved to find y_i at other intervals than boundary

(2) Shooting method

consider,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad y(a) = \alpha, \quad y(b) = \beta$$

Begin with $y'(a) = m_0$, Then

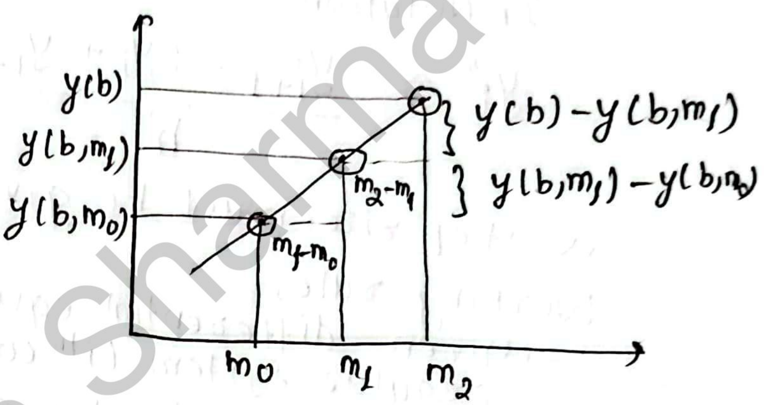
$$\left.\begin{aligned} \frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad y(a) = \alpha \\ y'(a) = m_0 \end{aligned}\right\} \underline{IVP} \rightarrow \text{Always use } \underline{RK4}$$

Solve this and obtain $y(b, m_0)$

Again guess $y'(a) = m_1$ & obtain $y(b, m_1)$

From similar triangle find exact value $\underline{m_2}$,

$$\frac{y(b) - y(b, m_1)}{y(b, m_1) - y(b, m_0)} = \frac{m_2 - m_1}{m_1 - m_0}$$



→ use this to find y at given interval

→ initial guess,

$$x_0 \checkmark, \quad y_0 \checkmark, \quad z_0 = y'(x_0) = y_0'$$

$$\left[\frac{dy}{dx} = z = y'(x)\right] \left[\frac{dz}{dx} = \frac{d^2y}{dx^2} = g(x, y, z)\right]$$

solution to system of linear equations:

1. Gauss Elimination method

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

← system of equations ← Matrix form ← Augmented matrix form.

$$\Downarrow$$

$$\left[\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$$

Upper triangular form (By elementary row transformation)

$$\Downarrow$$

solve by back-substitution

← If any pivots a_1, b_2' or c_3'' is zero we interchange rows s.t. pivots $\neq 0$.

• Gauss elimination with pivoting, (to avoid inverson)

→ Partial pivoting: find largest available coefficient in column below the pivot element then interchange two rows.
(Numerically largest coefficient)
[only row interchange]

→ Total pivoting: both columns and rows for largest coefficient are searched in.
(interchange of columns & rows
so, change of order of unknowns)

← consistent → have sol^n
← Inconsistent → no solution

eg: $0z = 0$
→ Infinite sol^n
$0z = 20$
→ No solution

2. Gauss Jordan
Eliminates all non-diagonal elements producing a diagonal matrix form by row operator.

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \Rightarrow \left[\begin{array}{ccc:c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$$

4

3. LU Factorization / Triangular method $\Rightarrow A = LU$

(i) Do little's method : $L \rightarrow$ unit lower triangular
$U \rightarrow$ upper triangular

(ii) Crout's method : $L \rightarrow$ lower triangular
$U \rightarrow$ unit upper triangular

Do little

$\rightarrow AX = B$

$\rightarrow A = LU$

so,

$L[UX] = B$

V

$LV = B \xrightarrow[\text{Substitution}]{F/w}$ Find V

$\rightarrow UX = V \xrightarrow[\text{Substitution}]{\text{Back}}$ Find X

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Dolittle.

LU=A

$u_{11} = a_{11}$	$u_{12} = a_{12}$	$u_{13} = a_{13}$ (1)
$l_{21} = \frac{a_{21}}{u_{11}}$ (2)	$u_{22} = a_{22} - a_{12} l_{21}$	$u_{23} = a_{23} - a_{13} * l_{21}$ (3)
$l_{31} = \frac{a_{31}}{u_{11}}$	$l_{32} = \frac{a_{32} - l_{31} * u_{12}}{u_{22}}$	$u_{33} = a_{33} - a_{13} l_{31} - u_{23} l_{32}$ (4)

$u \rightarrow$ corr. a − Product of corr. row & column (all possible)

$l \rightarrow \frac{\text{corr. a − Product of corr row \& column (all possible)}}{\text{column diagonal}}$

L ✓ U ✓

check LU = A

Then $AX = B$

$LUX = B$

$UX = V \rightarrow V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$\cancel{Ex}$: $LV = B \rightarrow$ find V

$UX = V \rightarrow$ find X

4. Matrix inverse using Gauss Jordan

we have, $AX = B$

$\hookrightarrow X = A^{-1}B$

(I) $[A : I] = \left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$

$\Downarrow$

(II) $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & a_{11}' & a_{12}' & a_{13}' \\ 0 & 1 & 0 & a_{21}' & a_{22}' & a_{23}' \\ 0 & 0 & 1 & a_{31}' & a_{32}' & a_{33}' \end{array}\right]$

$\underbrace{\qquad\qquad}_{A^{-1}}$

Iterative Methods → Jacobi
→ Gauss-seidal
→ Power-method

5. Jacobi's (Gauss-Jacobi) Method

(I)
$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{--- (1)}$$

tolerable error (e)

with, $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$, $|c_3| > |a_3| + |b_3|$

(Diagonally dominant system)

(II)
$$\left.\begin{aligned} x_{i+1} &= \frac{1}{a_1}(d_1 - b_1y_i - c_1z_i) \\ y_{i+1} &= \frac{1}{b_2}(d_2 - a_2x_i - c_2z_i) \\ z_{i+1} &= \frac{1}{c_3}(d_3 - a_3x_i - b_3y_i) \end{aligned}\right\}$$

(III) First approx x_0, y_0, z_0 are given & found successive iterative values x_1, y_1, z_1

unless, $|x_i - x_{i+1}| < e$, $|y_i - y_{i+1}| < e$, $|z_i - z_{i+1}| < e$

Required roots are $x_{i+1}, y_{i+1}, z_{i+1}$

→ Method of simultaneous displacement

In absence of any better estimates take x_0, y_0, z_0 as zero.

6. Gauss–Seidal Method (Diagonally dominant form)
is same as gauss–Jacobi but in successive iterations most recent values are used.

$$x_1^{(k+1)} = \frac{(b_1 - a_{12}\, x_2^{(k)} - a_{13}\, x_3^{(k)})}{a_{11}}$$

$$x_2^{(k+1)} = \frac{(b_2 - a_{21}\, x_1^{(k+1)} - a_{23}\, x_3^{(k)})}{a_{22}}$$

$$x_3^{(k+1)} = \left(\frac{b_3 - a_{31}\, x_1^{(k+1)} - a_{32}\, x_2^{(k+1)}}{a_{33}}\right)$$

for $\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

7. Power Method
(Dominant / Absolutely largest Eigen values)

$$\underset{\text{Matrix}}{A}\,\underbrace{X = \overset{\text{scalar}}{\lambda}\, X}_{\text{vector}}$$

$$|A - \lambda I| = 0$$

(I) $A\, X^{(0)} \xrightarrow{*} Z^{(0)} \longrightarrow \lambda^{(1)} \underbrace{X^{(1)}}$ obtained by dividing $Z^{(0)}$ by its largest element i.e. $\lambda^{(1)}$

(X_0 → initial guess) $[1 \;\; 0 \;\; 0]$

(II) $A\, X^{(1)} \xrightarrow{*} Z^{(1)} \longrightarrow \lambda^{(2)} X^{(2)}$

⋮

General,

$$A\, X^{(i)} \xrightarrow{*} Z^{(i)} \longrightarrow \lambda^{(i+1)} X^{(i+1)}$$

Repeat until
$$X^{(i+1)} \approx X^{(i)}$$

where,
$\lambda^{(i+1)}$ → dominated Eigen value
$X^{(i+1)}$ → corresponding vector.

In Normal form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{\text{as}} \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Eigenvector

- Finding smallest. Eigen value & corresponding vector

→ If $AX = \lambda X$ is true, then

$$A^{-1} X = \frac{1}{\lambda} X \text{ is also true.}$$

→ So,

If $\underbrace{A^{-1} X}_{} = \lambda X$ ← dominant value of A^{-1}

Then,

$AX = \frac{1}{\lambda} X$ ← smallest value of A

* Curve fitting

1. Linear curve fitting

$$y = a + bx \quad \text{—(I)}$$

Normal equations

$$\left.\begin{aligned} \Sigma y &= na + b\Sigma x \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 \end{aligned}\right\} \text{—(II)}$$

Solve these to find a & b

2. Non-linear curve fitting

(1) Exponential (x in power of something) $\{y = ae^{bx},\ y = ab^x\}$

$$y = ae^{bx}$$

$$\log y = \log a + bx$$

$$Y = A + bx$$

Σ Σx

$$\left.\begin{aligned} \Sigma Y &= nA + b\Sigma x \\ \Sigma xY &= A\Sigma X + b\Sigma x^2 \end{aligned}\right\} \rightarrow A \rightarrow \log(a) \Rightarrow a = e^{(A)}$$

b

(2) Power function

$$y = ax^b$$

$$\log y = \log a + b \log x$$

$$Y = A + bX$$

(3) Polynomial function ~~of~~

say IInd order,

$$y = a + bx + cx^2 \quad \text{—(I)}$$

$*\Sigma$: $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$*\Sigma x$: $\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$*\Sigma x^2$: $\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

← solve to find a, b, c.

Interpolation

[Surendra Sharma
PUL078BCE178]

- For equally spaced

(a) Newton's Forward Interpolation Formula

$$f(x) = y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots + \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!}\Delta^n y_0$$

y_0 = initial value.
y = function.

$$\left[p = \frac{x - x_0}{h}\right]$$

$x \rightarrow$ given
$h \rightarrow$ common difference.

(b) Newton's Backward interpolation formula

$$f(x) = y_n = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots\dots + \frac{p(p+1)(p+2)\dots(p+(n-1))}{n!}\nabla^n y_n$$

$$\left[p = \frac{x - x_n}{h}\right]$$

* Central Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

(iii) Central difference Method

① Stirling's Formula

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$

$$+ \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1^2)(p^2-2^2)}{6!}\Delta^6 y_{-3} + \dots$$

- factorial
- odd p
- even p^2

- follow center line ------
- odd → avg of two sides
- even → that term only (center)

(ii) Bessel's formula → center line ~~~

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{(p-\frac{1}{2})p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \dots$$

ote: If $-\frac{1}{4} \le p \le \frac{1}{4}$ → Stirling's
If $\frac{1}{4} < p \le \frac{3}{4}$ → Bessel's

$p = \dfrac{x - x_0}{h}$ central value

• For unequally spaced interval

(a) Lagrange's Interpolation formula

NO x_0 NO x_1

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} * y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} * y_1$$

$$+ \cdots\cdots + \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} * y_n$$

Divided difference table

x	y	1st divided diff	2nd divided diff	3rd divided diff	4th divided diff
x_0	y_0				
		$[x_0, x_1]$			
			$[x_0, x_1, x_2]$		
x_1	y_1			$[x_0, x_1, x_2, x_3]$	
		$[x_1, x_2]$			
			$[x_1, x_2, x_3]$		$[x_0, x_1, x_2, x_3, x_4]$
x_2	y_2				
		$[x_2, x_3]$		$[x_1, x_2, x_3, x_4]$	
			$[x_2, x_3, x_4]$		
x_3	y_3				
		$[x_3, x_4]$			
x_4	y_4				

(b) Newton's divided difference interpolation formula

$$y = y_0 + (x - x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] +$$

+1

$$\underbrace{(x-x_0)(x-x_1)(x-x_2)}_{\text{differences}}\underbrace{[x_0, x_1, x_2, x_3]}_{\text{divided diff}} + \cdots$$

Cubic spline interpolation:

Natural cubic spline $\rightarrow M_0$ & $M_n = 0$

- $$f_i(x) = \frac{M_{i+1}}{6h_i}(x-x_i)^3 - \frac{M_i}{6h_i}(x-x_{i+1})^3 + \left[\frac{y_{i+1}}{h_i} - \frac{M_{i+1}h_i}{6}\right](x-x_i) - \left[\frac{y_i}{h_i} - \frac{M_i h_i}{6}\right](x-x_{i+1}) \quad -(1)$$

$\rightarrow$ cubic spline b/w x_i, y_i & x_{i+1}, y_{i+1}

- M_i & M_{i+1} $\rightarrow$ values of 2nd derivative at x_i & x_{i+1}

$i = 0, 1, \dots, n-1$

- $$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} \Delta^2 y_0 \\ \Delta^2 y_1 \\ \Delta^2 y_2 \\ \Delta^2 y_3 \end{bmatrix}$$

Reduced system for natural cubic splice

Numerical differentiation:- [Surendra Sharma]

F/W

$$\rightarrow \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \dots\right]$$

At $x = x_0$, $p = 0$

$$\left(\frac{dy}{dx}\right)_{(x_0)} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + \dots\right]$$

(even → -ve)

$$\rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \dots\right]$$

At $x = x_0$, $p = 0$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_0)} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}(\Delta^4 y_0) - \frac{5}{6}\Delta^5 y_0 + \dots\right]$$

B/W

$$\rightarrow \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2!}\nabla^2 y_n + \frac{(3p^2+6p+2)}{3!}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_n + \dots\right]$$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \dots\right]$$

(all +ve)

$$\rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \dots\right]$$

Max & Min.

$\rightarrow \frac{dy}{dp} = 0$ (upto 3rd terms)

$$\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + \left(\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}\right) = 0$$

Stirling's Formula

$$\rightarrow \left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-1}}{2}\right) + \cdots\right]$$

[only odd Δ] avg (1) $\left(-\frac{1}{6}\right)$ $\left(+\frac{1}{30}\right)$

$$\rightarrow \left(\frac{d^2y}{dx^2}\right)_{(x_0)} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} \cdots\right]$$

Bessel's Formula.

$$\left(\frac{dy}{dx}\right)_{(x_0)} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots\right].$$

Integration

* Newton's cote Quadrature Formula

$$I = \int_a^b y\,dx = \int_0^n y\,h\,dp$$

$$I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\right]$$

(1) Trapezoidal ($n=1$) $\rightarrow$ any n

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

(2) Simpson's 1/3 ($n=2$) $\rightarrow$ even 'n'

$$I = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\right]$$

even

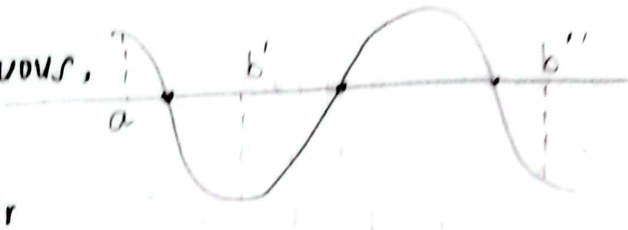
odd

(3) Simpson's 3/8 ($n=3$) — $n = *3$ — $*3$ — remai

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1})\right]$$

Solution of Non-linear Equations:

- Intermediate value theorem
  - → If a function $f(x)=0$ is continuous b/w an interval (a,b) such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one real root in the interval (a,b)
    - → (odd roots)



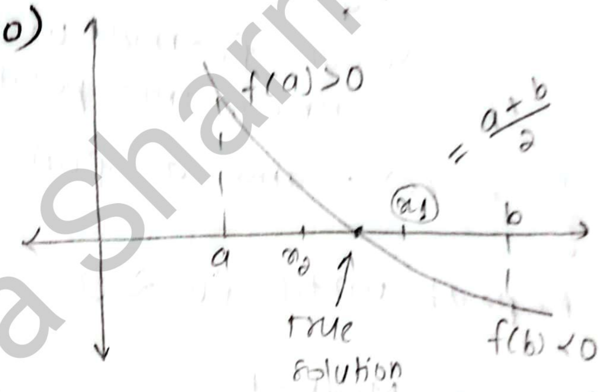
→ If function not-continuous, existence of root not guaranteed.

→ No roots or even number of roots if $f(a)$ & $f(b)$ of same sign.

(1) Bisection method (Bolzano)

↳ Divide & conquer

↳ repeated application of IVP



(i) choose a & b s.t. $f(a)>0$ & $f(b)<0$ or viceversa.

(ii) $x_1 = \dfrac{a+b}{2}$ & find $f(x_1)$

(iii) If $f(x_1)>0 \longrightarrow$ set $a=x_1$ & $b=b$

else, $\longrightarrow$ set, $a=a$ & $b=x_1$

- Stop?

  Say for accuracy of 3 decimal
  - → Tolerable error $(\varepsilon) = 0.0005$
  - → Absolute error : $|b-a| \le \varepsilon$
  - → Relative error: $\dfrac{|b-a|}{|a|} \le \varepsilon$ or $\dfrac{|b-a|}{|b|} \le \varepsilon$ or $\dfrac{|b-a|}{|c|} \le \varepsilon$
  - → % error = Relative error in %.
  - → Absolute error in function value, $|f(c)| \le \varepsilon$.

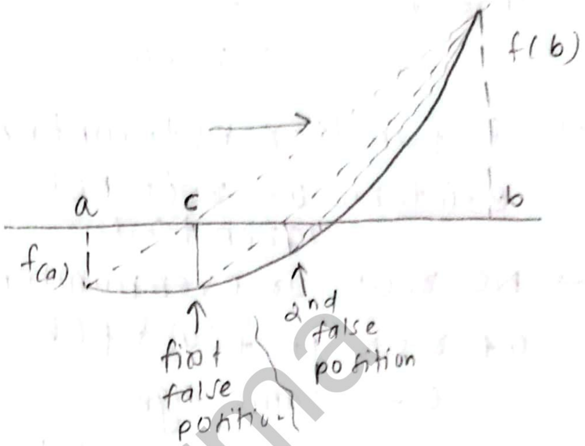
- No. of iterations, Error in n^{th} iteration $= \dfrac{|b-a|}{2^n}$

  After n iteration ⇒ $\dfrac{|b-a|}{2^n} \le \varepsilon$ ⇒ $n \ge \dfrac{\ln(|b-a|) - \ln\varepsilon}{\ln 2}$

Calculation table:

i	a (sign(+ or -ve))	b (sign(-ve or +ve))	$c = \frac{a+b}{2}$	$f(c)$

(2) Method of False Position (Regula Falsi Method)
↳ Solution much faster



$$\frac{f(a)}{a-c} = \frac{f(b)}{b-c}$$

$$\left[c = \frac{af(b) - bf(a)}{f(b) - f(a)} \right]$$

↓

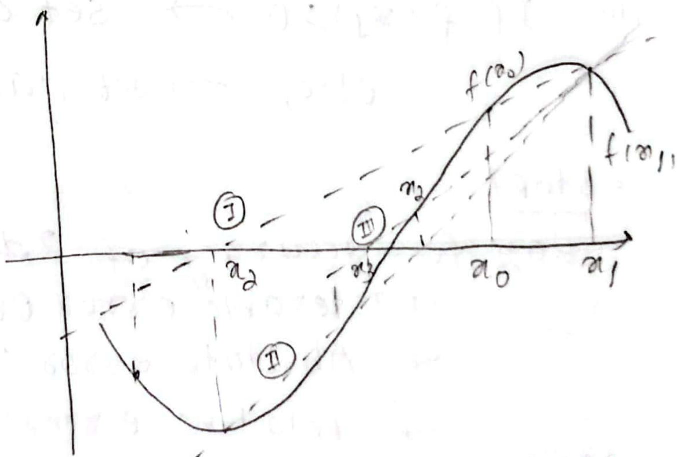
secant formula or linear interpolation formula.

{ If $f(c)$ same as $f(a)$ the move a to c
else move b to c

Repeat until $f(c) \approx 0$.

(3) Secant Method
↳ NO use of IVT
↳ convergence rate faster
↳ NO need to see functional value for discarding

Slope of secant line:



$$\frac{f(x_0)}{x_2 - x_0} = \frac{f(x_1)}{x_1 - x_2}$$

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$x_2 \to$ first approx value of root

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$\left[c = \frac{af(b) - bf(a)}{f(b) - f(a)} \right]$$

A = B : B = Rnd(C)

$c \to$ new value

$a \& b \to$ previous two old values.

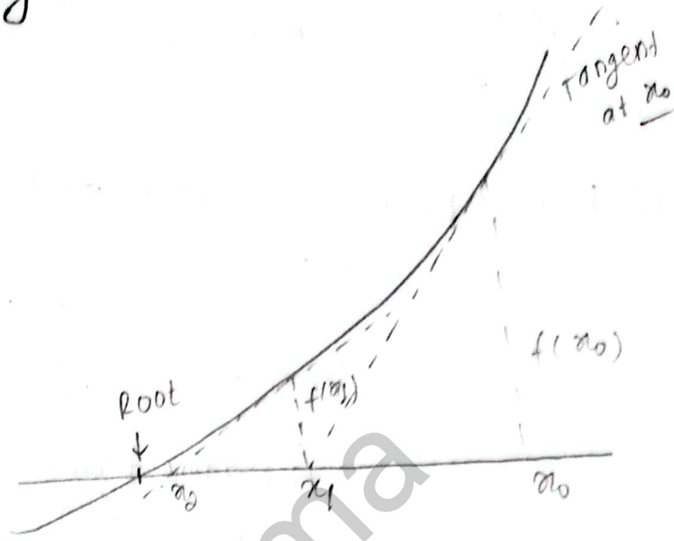
4. Newton-Raphson Method

(Tangent line approximation)

↳ Approximating curve by a straight line drawn on curve at a point very close to the root x_0.

$x_0 \rightarrow$ initial guess to root

$x_1 \rightarrow$ first approximation.



Slope of tangent at x_0,

$$= f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

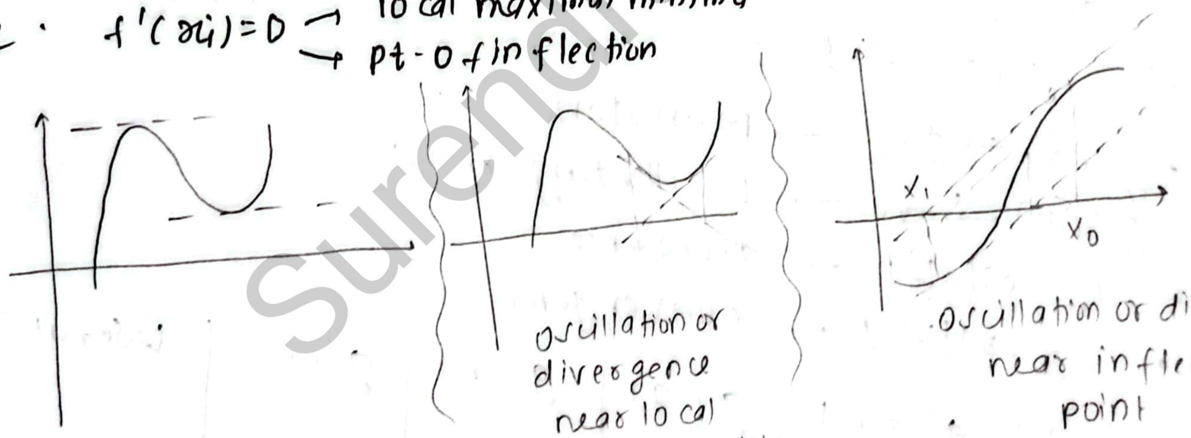
$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_1)}$$

$$\left[x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}\right]$$

↳ Repete until x_i & x_{i+1} nearly equal

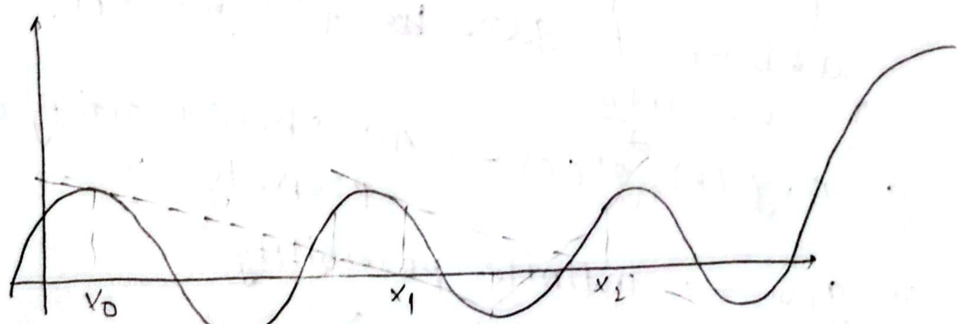
or, $f(x_i) \leq \varepsilon$.

* $f'(x) \rightarrow$ must (analytically)
* $f'(x_i) \rightarrow 0 \Rightarrow x_{i+1} \rightarrow \infty$
* $f'(x_i) = 0$ → local maxima/minima
  → pt. of inflection



oscillation or divergence near local maxima/minima

oscillation or divergence near inflection point



Root jumping (Multiple root curve)

Taylor series expansion

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots$$

$\dfrac{d(\log_{10} x)}{dx} = \dfrac{1}{x}\log_{10} e$

(5) Fixed point Iteration Method

$$\left.\begin{array}{c} f(x)=0 \\ \Downarrow \\ x=g(x) \end{array}\right\} \text{same } x \text{ satisfies both.}$$

$\Downarrow$

Iteration formula:

$x_{i+1} = g(x_i)$

(until x is constant)

* convergence properties

(i) $0 < g'(x) < 1 \Rightarrow$ Monotonic convergence

(ii) $g'(x) > 1 \Rightarrow$ Monotonic divergence

(iii) $-1 < g'(x) < 0 \Rightarrow$ oscillating/spiral convergence

(iv) $g'(x) < -1 \Rightarrow$ " " divergence

- $g'(x) \rightarrow \underline{+ve} \rightarrow$ Monotonic
- $g'(x) \rightarrow -ve \rightarrow$ oscillating/spiral
- $|g'(x)| > 1 \rightarrow$ divergence
- $|g'(x)| < 1 \rightarrow$ convergent

$x = g(x)$	$g'(x)$	$g'(a)$	$g'(b)$	$g'(c)$	useful?

$\begin{cases} a \,\&\, b \rightarrow \text{s.t. } \cancel{\text{good bar}}\ f(x) \text{ has opposite sign} \\ c = \dfrac{a+b}{2} \end{cases}$

$g'(a), g'(b), g'(c) \rightarrow$ All should satisfy convergence property.

$x = g(x) \rightarrow$ infinite possibility

- square root? $\left(x = \dfrac{\frac{N}{x} + x}{2} \right)$

PDE

[Surendra sharma]

2nd degree (linear PDE)
$$\left[A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G\right] \quad \text{— (1)}$$

→ Differential equation having more than one independent variable

→ If $G = 0$ → homogenous PDE

- Elliptic if $B^2 - 4AC < 0$
- Parabolic if $B^2 - 4AC = 0$
- Hyperbolic if $B^2 - 4AC > 0$

→ steady-state problem

(1) Elliptic equation

(i) Laplace eqn (2-D): $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ i.e. $\nabla^2 u = 0$

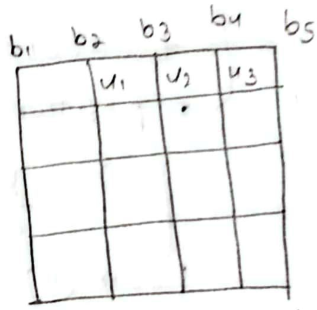
(ii) Poisson eqn: $u_{xx} + u_{yy} = f(x,y)$

Note:

Finite difference approximation,

$$y_i' \approx \frac{1}{2h}(y_{i+1} - y_{i-1})$$

$$y_i'' \approx \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$



use boundary condition to find $(u_1, \ldots, u_9)$

(PD)

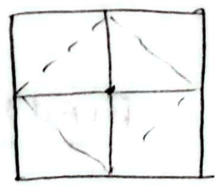
$$\frac{\partial u}{\partial x} = u_x \approx \frac{1}{2h}(u_{i+1,j} - u_{i-1,j})$$

$$\frac{\partial u}{\partial y} = u_y \approx \frac{1}{2k}(u_{i,j+1} - u_{i,j-1})$$

$$u_{xx} \approx \frac{\partial^2 u}{\partial x^2} \approx \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$u_{yy} \approx \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})$$

Then,
$$u_{xx} + u_{yy} = 0$$
[If square grid, $h = k$]



Thus,

standard five point formula (SFPF)
$$\left[u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}\right]$$

↳ Recurrence relation for laplace eqn.

• Poisson's equation:-

$u_{xx} + u_{yy} = f(x,y)$

$$\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \right]$$

[(i,j) coordinate का उपर left + right + up + down - 4 * त्यहाँ ॥ function $\times h^2$]

$[x = ih$ & $y = jh]$

coordinates No. of steps $\times$ step size

(2) One dimensional heat equation (Parabolic function)

$$\left[\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{i.e.} \quad u_t = c^2 * u_{xx} \right] \text{ - (ii)}$$

$u(x,t) \rightarrow$ tempr at any pt. of rod whose distance from left end of a thermally insulated rod after t seconds of heat conduction be x.

$\rightarrow$ position x & time t.

$\left[c^2 = \frac{k}{s\rho} \right] \rightarrow$ diffusivity of substance (cm²/sec)

From (ii),

$$\frac{1}{k}(u_{i,j+1} - u_{i,j}) - c^2 * \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] = 0$$

f/w

step-size for time

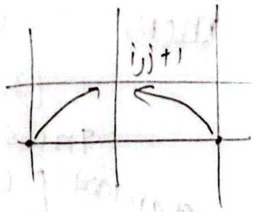
$$u_{i,j+1} = u_{i,j} + \frac{kc^2}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

Let, $\frac{kc^2}{h^2} = \alpha$

$$\left[u_{i,j+1} = (1 - 2\alpha) u_{i,j} + \alpha (u_{i+1,j} + u_{i-1,j}) \right.$$

$$\left. \underline{0 \le \alpha \le 1} \right]$$

Bendre-Schmitd method ($\alpha = 1/2$)



$$\left\{ u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}] \right\}$$

(j+1) time generation को value चाहियमा jth generation मा जाने व rt. व left को value को avg. जान्छ.

Initial condition, (initial temp^r at all pts of rod)

$$u(x,0) = u(x) \; ; \; 0 \le x \le L$$

Boundary condition: (temp^r at each end of rod as f^n of times)

$$u(0,t) \quad \& \quad u(L,t)$$

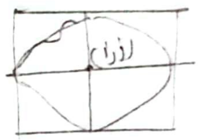
$$u(0,t) = c_1 \; , \; 0 \le t < \infty$$
$$u(L,t) = c_2 \; , \; 0 < t < \infty$$

$h \rightarrow$ step-size along length of rod.

- $u_t = c^2 u_{xx}$
- $\alpha = \dfrac{kc^2}{h^2}$
- $x = ih$
- ~~y~~ $t = jk$

$\Big]$ $i = 0, 1, 2 \ldots$; $j = 0, 1, 2, \ldots$

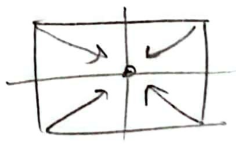
SFPF $\rightarrow$ $u_{i,j}$ = Avg. of four adjoining mesh points to left, right, above & below



DFPF (Diagonal five-point formula)

$$\hookrightarrow u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right]$$

(Avg. of four adjoining diagonal mesh points)



** DFPF $\rightarrow$ less accurate than SFPF
$\hookrightarrow$ to find starting values of mesh points.